

# Dirac fields, torsion and Barbero-Immirzi parameter in Cosmology

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**Abstract.** We consider cosmological solution for Einstein gravity with massive fermions with a four-fermion coupling, which emerges from the Holst action and is related to the Barbero-Immirzi (BI) parameter. This gravitational action is an important object of investigation in a non-perturbative formalism of quantum gravity. We study the equation of motion for the Dirac field within the standard Friedman-Robertson-Walker (FRW) metric. Finally, we show the theory with BI parameter and minimally coupling Dirac field, in the zero mass limit, is equivalent to an additional term which looks like a perfect fluid with the equation of state  $p = w\rho$ , with  $w = 1$  which is independent of the BI parameter. The existence of mass imposes a variable  $w$ , which creates either an inflationary phase with  $w = -1$ , or assumes an ultra hard equation of states  $w = 1$  for very early universe. Both phases relax to a pressure less fluid  $w = 0$  for late universe (corresponding to the limit  $m \rightarrow \infty$ ).

**Keywords:** Dirac fields, Cosmology, Barbero-Immirzi parameter

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## 1 Introduction

The investigation of Dirac fields in curved spacetime has been addressed in several works for many years, especially in the last decade (see, for example, Refs. [1–10]). There are many papers considering not only the classical aspects of the theory, but also quantum ones. In the present work, we deal with the classical aspects of massless Dirac particles in the spatially flat Friedmann-Robertson-Walker background in the presence of the Barbero-Immirzi parameter [11, 12], refers to as BI parameter in the following. This parameter, described by the Holst action term [13], was introduced under the non-perturbative quantum gravity perspective, and it represents then a new dimensionless parameter coming from a more fundamental theory.

The vacuum gravitational action together with Holst term can be written as

$$S_H = \frac{1}{\kappa} \int d^4x \sqrt{-g} \left\{ -R + \frac{1}{\beta} \epsilon_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right\}, \quad (1.1)$$

where  $\kappa = 16\pi G$ ,  $\epsilon_{\alpha\beta\mu\nu}$  is the Levi-Civita tensor and  $\beta$  is the BI parameter, which should be real and positive. In General Relativity (GR) this parameter makes no effect on dynamical equations, because  $\epsilon_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$  vanishes due to the cyclic symmetry of the Riemann tensor,  $R^{\alpha\beta\mu\nu} + R^{\alpha\nu\beta\mu} + R^{\alpha\mu\nu\beta} = 0$ . However, this is not so if torsion is present. The simplest way to introduce torsion is to consider the Einstein-Cartan action coupled to a Dirac field, such that the equations of motion for torsion become non-trivial. For an introduction and review on the theories with torsion, one can see [14–16], the last reference gives nice overview of the relevant cosmological models. It is worth noticing that, in this scenario, where fermionic matter is present the BI parameter and/or torsion can affect the gravitational dynamics, providing an interesting way to investigate its classical and quantum effects. It was first proposed in Ref. [17] that the Immirzi parameter leads to an effective four-fermion interaction mediated by the Immirzi parameter. The same effect can be achieved in the framework of Einstein-Cartan theory with minimal or nonminimal coupling of fermions to torsion.

The first studies of the effect of the Immirzi parameter was done using the minimal coupling procedure (MCP) of the fermions to gravity [17, 18]. However it was pointed out first in [18] that the effective four-fermions interaction depends on the choice of the coupling of fermions to gravity and that we can assume non minimal coupling procedure.

This is due to the fact that the action describing Dirac theory in flat spacetime is invariant by the introduction of a total divergence, say,  $\partial_\mu V^\mu$  with  $V^\mu = \bar{\psi}\gamma^\mu\psi$  or  $V^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ . However when torsion is present, there may be an extra coupling, for such terms produces extra term  $T_\mu V^\mu$  in the action (here  $T_\mu = T_{\mu\alpha}^\alpha$  is the trace of the torsion tensor). It means that a whole class of equivalent actions in flat spacetime are related (though MCP) to inequivalent actions in curved spacetime. This problem was especially addressed by Kibble [19] (see also [15]).

In [18] a one complex parameter family of non minimal coupling was investigated, in [20] another family of non-minimal coupling was proposed while in [21] a general 3 parameter family of non minimal coupling was investigated. In our note we *assume* that the fermion coupling is the minimal coupling and we investigate the consequences of this coupling on the cosmological evolution.

It can be argued that the minimal coupling is the most natural coupling of gravity to fermion based on the fact that non minimal couplings are sourced by components of the torsion that do not appear naturally in models of spinning matter. For instance it is possible to treat fermions, at a large scale, as a fluid with an intrinsic spin density. Following the Lagrangian formalism for this exotic spin fluid [22–24], one can see that the contribution from such fluid always produces a traceless torsion [25] (for the extensive discussion of the theories with spin fluid in cosmology, see also Refs. [26–32]). In this case the trace of the torsion tensor  $T_\mu = T_{\mu\alpha}^\alpha$  has smaller physical meaning and its presence is completely due to the nonminimal procedure [14, 15]. It turns out that the non-minimal procedure [33]

$$\partial_\mu \rightarrow \nabla_\mu + i\eta_1 \gamma^5 S_\mu + i\eta_2 T_\mu \quad (1.2)$$

(we use the notations of [15], correspondingly  $\nabla_\mu$  is the Riemannian covariant derivative, without torsion) is the most consistent one, especially at the quantum level. Of course it is possible to view this procedure as a MCP with a modified connection [34]. In any case, the result will be the equivalent action with current-current interaction found in [17]. For the corresponding discussion of the issue related to the BI parameter effects, see, for example, Refs. [34–36].

In the present paper we consider a particular cosmological solution, with a FRW-like metric, within the standard approach to BI parameter, and show that this parameter does affect the cosmological solution.

The paper is organized as follows. In the next section we briefly discuss the existing ambiguity of the MCP in Einstein-Cartan theory. In Sect. 3 the details of the derivation of equations for the metric and for the components of the Dirac field are given. The details of the derivation of the Energy-Momentum Tensor are settled in the Appendix A. In Sect. 4 we discuss the cosmological solution and verify the consistency conditions. The calculations are not completely trivial and hence we present them in some details. Finally, in Sect. 5 we draw our conclusions and discuss possible perspectives of cosmology based on the gravity theory with BI parameter.

## 2 Derivation of dynamical equations

The action for gravity theory with an additional Holst term with the BI parameter and minimal fermion coupling, in the presence of massive Dirac fields can be written in terms of

a spin-spin coupling in the form [17, 18]

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{\kappa} R + \frac{i}{2} (\bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi - \theta J_\mu J^\mu \right\}, \quad (2.1)$$

where  $\theta$  is the spin-spin parameter related to Immirzi parameter  $\beta$  by

$$\theta = \frac{3\kappa}{32} \frac{\beta^2}{\beta^2 + 1} \quad (2.2)$$

and the axial current is defined as

$$J^\mu = \bar{\psi} \Gamma_5 \gamma^\mu \psi. \quad (2.3)$$

Here,  $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$  is the chiral Dirac matrix. The Dirac matrices,  $\gamma^\mu$ , are expressed in terms of the vierbeins  $e^a{}_\nu$  and the Dirac matrices  $\Gamma^a$  in tangent flat space according to  $\gamma^\mu = e_a{}^\mu \Gamma^a$ .

Two important observations are in order. The same action (2.1) can be seen as the result of integrating out completely antisymmetric torsion in the Einstein-Cartan gravity coupled to the fermion fields forming axial current. In this sense all our further consideration can be attributed to one of the two models, namely Holst gravity with BI parameter and Einstein-Cartan gravity. Further aspects of the difference between these two formulations were discussed in [20].

The second observation concerns the relevance of the choice of the MCP procedure, which we have already mentioned in the Introduction. The MCP corresponds to the value  $\eta_1 = 1/8$  in (1.2). Taking into account the relation of our model with the Einstein-Cartan gravity, it becomes obvious that the transition to an arbitrary non-minimal interaction between fermion and torsion is performed by the multiplication of  $\theta$  in (2.2) by a factor of  $64\eta_1^2$ . Therefore the “nonminimal” value of the parameter should be

$$\theta_{\text{nonminimal}} = \frac{6\kappa \eta_1^2 \beta^2}{\beta^2 + 1}. \quad (2.4)$$

In the ideal world with only one kind of non-interacting fermions, this would be the end of the story, because the single parameter  $\theta$  contains the information about both  $\beta$  and  $\eta_1$ . However, in many physically interesting theories such as Standard Model of particle physics, there are different kinds of fermions, all of them interacting to scalar (Higgs) by means of Yukawa interaction. At quantum level the nonminimal parameters  $\eta_1$  for different fermions are renormalized, become running parameters, and their running depends on the values of Yukawa couplings [33]. As a consequence, the values of  $\eta_1$  must be different for different fermions. In what follows we present a simplified consideration with only one  $\theta$ , corresponding to the one single type of fermionic axial current  $J^\mu$ . One can regard this approach as simplification, which is based on the assumption that the running of different parameters  $\eta_1$  is not very strong and that MCP hypothesis can serve, after all, as a good approximation.

The variation with respect to the Dirac fields gives the dynamical equations

$$i\gamma^\mu \nabla_\mu \psi = 2\theta J^\mu \Gamma_5 \gamma_\mu \psi + m\psi, \quad (2.5)$$

$$i\nabla_\mu \bar{\psi} \gamma^\mu = -2\theta J^\mu \bar{\psi} \Gamma_5 \gamma_\mu - m\bar{\psi}. \quad (2.6)$$

The Einstein equations, coming from the variation with respect to  $g_{\alpha\beta}$ , can be written as<sup>1</sup>

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = G_{\alpha\beta} = \frac{\kappa}{2} T_{\alpha\beta} = \frac{i}{4} \kappa [\bar{\psi} \gamma_{(\alpha} \nabla_{\beta)} \psi - \nabla_{(\beta} \bar{\psi} \gamma_{\alpha)} \psi] - \frac{1}{2} \kappa g_{\alpha\beta} \mathcal{L}, \quad (2.7)$$

where

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi - \theta J_\mu J^\mu, \quad (2.8)$$

and  $\gamma_{(\alpha} \nabla_{\beta)}$  means  $\frac{1}{2}(\gamma_\alpha \nabla_\beta + \gamma_\beta \nabla_\alpha)$ . By using equation (2.5), we obtain

$$G_{\alpha\beta} = \frac{i}{4} \kappa [\bar{\psi} \gamma_{(\alpha} \nabla_{\beta)} \psi - \nabla_{(\beta} \bar{\psi} \gamma_{\alpha)} \psi] - \frac{\theta}{2} \kappa g_{\alpha\beta} J_\mu J^\mu. \quad (2.9)$$

Let us note that our expression for the energy-momentum tensor (2.7) is slightly different from the one of [1], which was obtained by variation with respect to the vierbein. The expression (2.7) has been obtained by varying with respect to the metric. The technical details can be found in the Appendix A.

The next step is to calculate the equations for the specific metric corresponding to the homogeneous and isotropic universe (FRW space-time). The spin connection,

$$\begin{aligned} \omega^a{}_{\mu}{}^b = \frac{1}{4} & \left( e^{b\alpha} \partial_\mu e^a{}_\alpha - e^{a\alpha} \partial_\mu e^b{}_\alpha + e^{a\alpha} \partial_\alpha e^b{}_\mu - e^{b\alpha} \partial_\alpha e^a{}_\mu + \right. \\ & \left. + e^{b\nu} e^{a\lambda} e_{c\mu} \partial_\lambda e^c{}_\nu - e^{a\nu} e^{b\lambda} e_{c\mu} \partial_\lambda e^c{}_\nu \right), \end{aligned} \quad (2.10)$$

can be obtained for the FRW spacetime with null spatial curvature,  $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2)$ , by taking into account the vierbeins

$$e^a{}_0 = (a(\eta), 0, 0, 0), \quad e^a{}_1 = (0, a(\eta), 0, 0), \quad e^a{}_2 = (0, 0, a(\eta), 0), \quad e^a{}_3 = (0, 0, 0, a(\eta)) \quad (2.11)$$

Here we consider the conformal time,  $\eta$ , defined by  $a(\eta)d\eta = dt$ . Thus, the non-null spin connection components  $\omega^a{}_{\mu}{}^b$  are

$$\omega^{01}{}_1 = \omega^{02}{}_2 = \omega^{03}{}_3 = \frac{a'}{2a}, \quad \omega^{10}{}_1 = \omega^{20}{}_2 = \omega^{30}{}_3 = -\frac{a'}{2a}, \quad (2.12)$$

where  $a' = da/d\eta$ .

### 3 Cosmological solution and consistency conditions

#### 3.1 Dirac equation

In order to solve the equation in a cosmological background, we make the assumption that the fermions are spatially constant,  $\partial_i \psi = 0$ . With this assumption and using the form of the connection (2.12) we get

$$\nabla_0 \psi = \partial_0 \psi, \quad \nabla_0 \bar{\psi} = \partial_0 \bar{\psi}, \quad \nabla_i \psi = \frac{a'}{2a} \Gamma_i \Gamma_0 \psi, \quad \nabla_i \bar{\psi} = \frac{a'}{2a} \bar{\psi} \Gamma_0 \Gamma_i, \quad i = 1, 2, 3. \quad (3.1)$$

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<sup>1</sup>The expression for the Energy-Momentum Tensor for the free case is known from [37].

The Dirac equations (2.5) and (2.6) can be written as

$$\begin{aligned} \frac{i}{a} \Gamma_0 \left( \psi' + \frac{3a'}{2a} \psi \right) &= m\psi + 2\theta (\Gamma_5 \Gamma_a \psi) J^a \\ -\frac{i}{a} \left( \bar{\psi}' + \frac{3a'}{2a} \bar{\psi} \right) \Gamma_0 &= m\bar{\psi} + 2\theta J^a (\bar{\psi} \Gamma_5 \Gamma_a). \end{aligned} \quad (3.2)$$

where  $J_a \equiv (\bar{\psi} \Gamma_5 \Gamma_a \psi)$ .

From these equation we can extract the conservation law:

$$(\bar{\psi} \Gamma_0 \psi)' = -\frac{3a'}{a} (\bar{\psi} \Gamma_0 \psi). \quad (3.3)$$

which is identical to the current conservation

$$0 = \nabla_\mu (\bar{\psi} \gamma^\mu \psi) = \partial_\mu (\bar{\psi} \gamma^\mu \psi) + \Gamma^\mu_{\rho\mu} (\bar{\psi} \gamma^\rho \psi), \quad (3.4)$$

where  $\Gamma^\mu_{\rho\mu}$  is the Riemannian affine connection. This can be easily integrated out

$$V_0(\eta) = \frac{M^3}{a^3}$$

where  $M$  is a mass scale that gives the size of the condensate and  $V_a \equiv \bar{\psi} \Gamma_a \psi$  is the vectorial current.

One can also get the following equations for the symplectic current:

$$\frac{i}{2a} (\bar{\psi} \Gamma_0 \psi' - \bar{\psi}' \Gamma_0 \psi) = 2\theta J_a J^a + m\bar{\psi} \psi, \quad (3.5)$$

$$\frac{i}{2a} (\bar{\psi} \Gamma_i \psi' - \bar{\psi}' \Gamma_i \psi) = 0. \quad (3.6)$$

In order to analyze further the spinor equations of motion, let us introduce not only the vectorial and axial current  $V_a$  and  $J_a$  but also the pseudo scalar  $P \equiv i\bar{\psi} \Gamma_5 \psi$ , scalar  $S \equiv \bar{\psi} \psi$  and Lorentz tensor  $L_{ab} = \frac{i}{2} \bar{\psi} [\Gamma_a, \Gamma_b] \psi$  currents.

We can check that the equation of motion for the scalar  $S$  and pseudo scalar  $P$  and the time component of the axial current form a closed subset of equation. Indeed, in the Appendix B we derived the closed set of equations

$$\frac{1}{2a} (a^3 S)' = 2a^3 \theta P J_0, \quad (3.7)$$

$$\frac{1}{2a} (a^3 J_0)' = -a^3 m P, \quad (3.8)$$

$$\frac{1}{2a} (a^3 P)' = a^3 (m J_0 - 2\theta S J_0). \quad (3.9)$$

From these equation we can conclude that  $a^6 (S^2 + J_0^2 + P^2)$  is a conserved quantity. Moreover we have the continuity equation:

$$[a^3 (\theta J_0^2 + m S)]' + 3a' a^2 \theta J_0^2 = 0. \quad (3.10)$$

We can also show that the spatial components of the axial vector are conserved,

$$(a^3 J_i)' = 0. \quad (3.11)$$

For completeness we can also give the evolution equation for the other currents:

$$\begin{aligned}\frac{1}{2a^4} (a^3 V_i)' &= m L_{0i} + 2\theta \epsilon_{iaj} J^a V^j, \\ \frac{1}{2a^4} (a^3 L_{0i})' &= -m V_i + 2\theta \epsilon_{iab} J^a L^{0b} - \theta J_0 \epsilon_{ijk} L^{jk}, \\ \frac{1}{2a^4} (a^3 L^i)' &= 2\theta (\epsilon^{jki} J_j L_k - 2J_0 L^{0i}),\end{aligned}\tag{3.12}$$

where  $L^i \equiv 1/2 \epsilon^{ijk} L_{jk}$ , and  $\epsilon_{aij}$  is the 3-dimensional Levi-Civita tensor ( $\epsilon_{aij} = 0$  if  $a = 0$ ). One can extract from these equations one conserved quantity, namely

$$a^6 (V_i V^i + L^{0i} L_{0i} + L_i L^i).\tag{3.13}$$

### 3.2 Einstein equation

The spatial and time-space off-diagonal components of the fermion energy momentum tensor are given by

$$T_{ij} = \frac{i}{2} (\bar{\psi} \gamma_{(i} \nabla_{j)} \psi - \nabla_{(j} \bar{\psi} \gamma_{i)} \psi) + a^2 \eta_{ij} \theta J^\mu J_\mu = 0, \quad i \neq j,\tag{3.14}$$

$$T_{0i} = \frac{i}{2} (\bar{\psi} \gamma_{(0} \nabla_{i)} \psi - \nabla_{(0} \bar{\psi} \gamma_{i)} \psi),\tag{3.15}$$

$$T_{00} = \frac{i}{2} (\bar{\psi} \gamma_0 \nabla_0 \psi - \nabla_0 \bar{\psi} \gamma_0 \psi) - a^2 \theta J^\mu J_\mu.\tag{3.16}$$

Direct algebraic manipulation with Eq. (3.14), using the derivative (3.1) together with the identity  $\Gamma_{(i} \Gamma_{j)} \Gamma_0 - \Gamma_0 \Gamma_{(j} \Gamma_{i)} = 0$  lead to the conclusion that the first term in (3.14) vanishes. One can also easily see that the condition (3.15) is proportional to the LHS of (3.6) and thus also vanish. Finally, thanks to (3.5) and (3.1) one can see that the first term of (3.16) is equal to  $a^2 (2\theta J_a J^a + m \bar{\psi} \psi)$ .

In summary this shows that one can express the energy-momentum tensor of the fermion in the standard perfect-fluid form,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu},\tag{3.17}$$

where, for the FRW metric,  $u_\mu = (a, 0, 0, 0)$ . Using identifications  $T_{00} = \rho a^2$  and  $T_{ij} = p a^2 \eta_{ij}$ , we are able to express  $\rho$  and  $p$  as<sup>2</sup>

$$\rho = \theta J^2 + m S \quad \text{and} \quad p = \theta J^2.\tag{3.18}$$

where  $J^2 = J_\mu J^\mu = J^a J_a$ .

The energy conservation law reads

$$\dot{\rho} + 3H(\rho + p) = 0 \quad \text{with} \quad H = \dot{a}/a,\tag{3.19}$$

This is consistent with the Dirac equations of motion derived previously.

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<sup>2</sup>Let us note that the zero Energy-Momentum tensor without  $\theta$  in the massless case is only due to the fact we perform this calculation for a conformally-flat metric. In this case the trace of the Energy-Momentum tensor completely controls all its components. Some extended discussion of this issue was recently given in [38].

For the FRW spacetime, every off-diagonal component of Einstein tensor vanishes identically,  $G_{0i} = G_{ij} = 0$  ( $i \neq j$ ). This is consistent with the form of the energy-momentum tensor just calculated above. The temporal and spatial components of Einstein equations (2.9) read, respectively,

$$\frac{3a'^2}{a^2} = \frac{\kappa}{2}a^2(\theta J^2 + mS) = \frac{\kappa}{2}a^2\rho, \quad (3.20)$$

$$-\frac{2a''}{a} + \frac{a'^2}{a^2} = \frac{\kappa}{2}a^2\theta J^2 = \frac{\kappa}{2}a^2p, \quad (3.21)$$

while taking the trace of the Einstein equation, we can express the acceleration in terms of the physical time coordinate  $t$  as

$$\frac{6\ddot{a}}{a} = -\frac{\kappa}{2}(4\theta J^2 + mS). \quad (3.22)$$

### 3.3 Massless case

In the case where  $m = 0$  we can see that the BI parameter for a massless Dirac field produces the same effect as of a perfect fluid characterized by the equation of state (EOS)  $p = \rho = \theta J^2$ . It is remarkable that, if  $m = 0$ , both energy density and pressure in eq. (3.18) do depend on  $\theta$ , while their ratio does not. Hence the theory under consideration does not have a smooth limit for  $\theta \rightarrow 0$ . In this case the energy conservation law (3.19) gives us a new scaling law for energy density of massless fermions in a gravity theory with non-zero BI parameter,

$$\rho \sim a^{-6}. \quad (3.23)$$

In the considerations presented above we did not take into account the energy density and pressure of the free part of the fermion spinor field. According to our calculations this part is zero. However, this apparent result is only due to our specific treatment of massless fermionic field in the cosmological setting, where we are essentially looking for the non-conformal part of  $T_{\mu\nu}$ . The conformal part which does not couple to the conformal factor  $a(t)$  can not be seen in this approach for the following reason. We were looking for the time-dependent spinor field, but this is definitely not a right idea for the massless field which can not be done space-independent by any choice of the local reference frame. In order to see this, in the conformal case, it is sufficient to consider the flat-space limit. The space-independent fermion satisfies the equation

$$\Gamma^0 \partial_0 \psi = 0, \quad (3.24)$$

that simply means  $\partial_0 \psi = 0$  and constant field. Then the Eqs. (3.14), (3.15) and (3.16) tell us that  $T_{\mu\nu} = 0$ . However the right way to take the energy density and pressure in the conformal part of fermion  $T_{\mu\nu}$  into account is quite different. One has to remember that in the massless field case the dependence on time and space coordinates should be related, because the velocity is universally fixed for all modes of the field. For example, the free solution of  $i\Gamma^\mu \partial_\mu \psi = 0$  can be always considered as a superposition of massless plane waves

$$\psi(x^0, x^i) = v e^{i(k_0 x^0 - \mathbf{k}\mathbf{r})}, \quad k_0^2 - \mathbf{k}^2 = 0. \quad (3.25)$$

It is an elementary exercise to check that each one of these waves creates pressure which is 1/3 of its energy density. Therefore the total pressure  $p_0$  and energy density  $\rho_0$  of the



free fermionic field are related as  $p_0 = \rho_0/3$ . Taking into account our previous result we arrive at the following energy density and pressure of the fermion field in the presence of BI parameter:

$$\rho = \rho_0 + \theta J^2 \quad \text{and} \quad p = p_0 + \theta J^2 = \frac{1}{3} \rho_0 + \theta J^2. \quad (3.26)$$

Let us remark that the energy density and pressure of the free radiation content of the Universe is not necessary fermionic one, but can also include other fields, such as electromagnetic radiation. The unique assumption which was done here is that these two quantities are related by the equation of state  $p_0 = \rho_0/3$ , and this is indeed satisfied for the asymptotically free ultra-relativistic fields independent on their spin.

The Eqs. (3.26) indicate that the matter content of the early Universe in the presence of the Holst term is characterized by a sum of two fluids, one of them is conventional radiation and another has equation of state which is independent on the value of  $\eta$ . Of course, the relevance of this second component depends on  $\theta$ , but the equation of state does not. In order to have better understanding of the new terms with  $\theta J^2$  in (3.26), we can consider the simplified cosmological model with vanishing  $\rho_0$  and  $p_0$ . It is easy to solve the Friedmann equation,  $H^2 = (8\pi G/3)\rho$  and obtain the corresponding rule for the expansion of the universe,

$$a \sim t^{1/3}, \quad (3.27)$$

which, again, is different from the laws of expansion for free radiation (with  $a \sim t^{1/2}$ ) and dust (with  $a \sim t^{2/3}$ ). The results (3.23) and (3.27) do not depend on the value of BI parameter. The only one important requirement is that this parameter should be nonzero. However, the scaling law (3.23) shows that the new terms decay much faster than radiation and hence become irrelevant in the course of expansion of the Universe. At the same time, there is a chance to see the traces of such term in observational cosmological data, especially in the ones related to the cosmic perturbations. In order to elaborate this idea one has to develop the complete cosmological model with BI parameter. We postpone this issue for the future work.

### 3.4 Massive case

In the massive case one has to distinguish the early and late time regime. Let us recall that according to the Dirac equations we have the conservation rule

$$J_0^2 + P^2 + S^2 = \frac{M^6}{a^6}, \quad J_i^2 = \frac{\tilde{M}^6}{a^6} \quad (3.28)$$

where  $M, \tilde{M}$  are mass scales that control the size of the fermion condensate. This shows that  $S$  is necessarily decreasing as the universe expand, The late regime of the system is attained when

$$a^3 \gg \frac{2\theta M^3}{m}.$$

In this regime we have that  $2\theta S \ll m$  which means that we can neglect the non linear term in Eq. (3.9) and the equations for the evolution of  $J_0, S, P$  are dominated by the mass term; the influence of the  $\theta$  coupling can be ignored. We have, in this case,

$$\partial_t(a^3 S) \approx 0, \quad (3.29)$$

$$\partial_t(a^3 J_0) \approx -2m a^3 P, \quad (3.30)$$

$$\partial_t(a^3 P) \approx 2m a^3 J_0. \quad (3.31)$$

This system of equation can be easily solved to give

$$J_0 = \frac{M_1^3}{a^3} \cos [2m(t - t_0)] , \quad P = \frac{M_1^3}{a^3} \sin [2m(t - t_0)] , \quad S = \frac{M_2^3}{a^3} , \quad (3.32)$$

where  $M_1$  and  $M_2$  are mass scales characterizing the condensate and satisfying  $M^6 = M_1^6 + M_2^6$ . In this late time evolution the condensate evolve like a pressure less fluid with  $p = 0$  where the energy density is dominated by the scalar component and evolve like  $\rho \sim a^{-3}$  while  $a \sim t^{2/3}$ .

Another way to understand this result comes from the fact that the energy density  $\rho = \theta J^2 + mS$  possess two components. The scalar components scales like  $S \sim a^{-3}$  while the current component scales like  $J^2 \sim a^{-6}$  therefore at late time the energy contribution is dominated by the scalar component.

On the other hand this means that at an earlier time the current is going to dominate the dynamic of the cosmological evolution. The crossover time takes place when we can no longer neglect the influence of the non linear  $\theta$  term in the fermionic evolution equation (3.9), that is when

$$a^3 \sim \frac{2\theta M^3}{m} . \quad (3.33)$$

At this crossover time both  $mS$  and  $\theta J^2$  are of the same order of magnitude provided we assume that  $M_1, M_2$  and  $\tilde{M}$  are of comparable magnitude. The energy density at this time is of the order  $m^2/\theta$  that is of the order  $(\beta^2 + 1)/\beta^2 m^2 M_P^2$  where  $M_P$  is the Planck mass.

There are then two radically different early time evolution depending on whether at this cross over time the current is space like or timelike. Let us assume first that at this cross-over time the current is timelike: i-e  $J_0^2 - J_i^2 > 0$  and that  $\theta J^2$  is of the same order as  $mS$ . In this case the energy density is dominated at an earlier time by the current term  $\theta J^2$ , the fermionic field becomes effectively massless and the dynamic for the fermionic equation of motion is entirely dominated by the non linear term and given by the massless equations:

$$\begin{aligned} \frac{1}{2a}(a^3 S)' &\approx 2\theta J_0(a^3 P) , \\ \frac{1}{2a}(a^3 J_0)' &\approx 0 , \\ \frac{1}{2a}(a^3 P)' &\approx -2\theta J_0(a^3 S) . \end{aligned} \quad (3.34)$$

In this regime applicable for early time

$$a^3 \ll \frac{2\theta M^3}{m} , \quad (3.35)$$

the solution reads

$$J_0 = \frac{M_1^3}{a^3} , \quad S = \frac{M_2^3}{a^3} \sin \left( \int_{\eta'_0}^{\eta} \frac{2\theta M_1^3}{a^3} \right) , \quad P = \frac{M_2^3}{a^3} \cos \left( \int_{\eta'_0}^{\eta} \frac{2\theta M_1^3}{a^3} \right) . \quad (3.36)$$

where  $M^6 = M_1^6 + M_2^6$ . In this early time regime the condensate evolves as massless condensate  $p = \rho$  with  $\rho \sim a^{-6}$  while  $a \sim t^{\frac{1}{3}}$ . It is quite remarkable that the early time evolution although dominated by the presence of  $\theta$  is following an equation of state independent of it.

Finally, there is another early time regime accessible to our system. This regime happens if  $J^2 < 0$ , i.e., the current is space-like at the cross-over time. In this case we cannot have that the current term  $\theta J^2$  dominates the scalar contribution since we should respect the constraint that  $\rho = \theta J^2 + mS \geq 0$ . What happens in this case is that the quantity  $\rho + p = 2\theta J^2 + mS$  which is positive at late time decrease to eventually vanish. When this quantity vanishes, and since  $\dot{\rho} = -3H(\rho + p)$ , the energy density becomes constant. This means that we enter an inflationary phase provided the value of the density energy is not zero. This shows that the other regime of the theory, characterized by a spacelike current at early time correspond to an inflationary era that relax to pressureless dust at late time. The value of the effective cosmological constant at early time depends on the initial condition but it can be bounded by the value of  $mS$  at crossover over time. Indeed, since the inflationary era is characterized by  $2\theta J^2 = -mS$ , the effective cosmological constant is  $\rho_\Lambda = mS/2$ . The value of  $mS/2$  at early time is necessarily smaller than its value at the cross over time which is  $m^2/2\theta^2$ . So

$$\rho_\Lambda \leq \frac{m^2}{4\theta^2}. \quad (3.37)$$

## 4 Conclusions

We have considered the cosmological solution for the metric-spinor gravity with the Holst term, especially the effect of the non-zero BI in a cosmological setting and shown that there are FRW-compatible solutions. One of the most remarkable result is that the EOS of the self-interacting spinor matter does not depend on the BI parameter in the massless limit, if we disregard the effect of free massless fermions. For the massive case we have identified two different regime. In the first regime the theory is effectively massless at early time and behaves as a perfect fluid with an ultra-hard equation of state  $w = 1$ . In the second regime the fermionic matter behaves effectively as a cosmological constant and creates an inflationary phase which is relaxed at late time into a pressureless fluid.

It would be definitely interesting to check whether the same effects take place for the spinning fluid. In the positive case this may lead to a potentially observable consequences for the early Universe. The effect of torsion and/or Holst term on the EOS for the hot matter is depending on the existence of axial current and on an arbitrary parameter  $\theta$  defined in Eqs. (2.2) or (2.4). In principle, some cosmological observations can be helpful in getting an upper bound for  $\theta$ . Let us note that recently the discussion of the effect of Holst term and/or torsion on the difference between EOS for photons and hot fermions (quarks and leptons) has been discussed in [41] in the framework of free fermion theory and loop quantum gravity. The motivation for this study was to see whether the fine balance required by Big Bang nucleosynthesis (BBN) holds in the presence of loop quantum gravity effects. It was shown that the possible violation of such a fine balance due to the possible effects of torsion or Holst term is very small. In our understanding, the real physical situation can be even more simple, because fermions and photons are supposed to be interacting with each other at the BBN epoch. Such an interaction can control the balance between fermions and radiation through emission and absorption of photons by fermions and also by quantum effects such as creation and annihilation of fermion-antifermion pairs by radiation. As a result, the EOS for the matter content of the Universe should be treated as unique at that epoch and the effect of loop quantum gravity or torsion can not probably change, in principle, the fine balance between expansion rates of fermions and radiation. At the same time, this total EOS can

be affected by the mentioned manifestations of a new physics, including the four-fermion interaction.

**Note added.** When we were preparing this manuscript for submission, the preprint with partially similar content [40] has been published.

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## Appendix A. Derivation of the Energy-Momentum Tensor

For this end one has to take

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad g'^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}, \quad \sqrt{-g'} = \sqrt{-g} \left(1 + \frac{h}{2}\right), \quad (4.1)$$

where only the contributions up to the first order in  $h^{\mu\nu}$  were kept, and  $h = h^\mu{}_\mu = h_{\mu\nu}g^{\mu\nu}$ . The expansion for the vierbeins are [39]

$$e'^{b\alpha} = e^{b\alpha} - \frac{1}{2}h^\alpha{}_\beta e^{b\beta}, \quad e'^a{}_\alpha = e^a{}_\alpha + \frac{1}{2}h_\alpha{}^\beta e^a{}_\beta, \quad (4.2)$$

such that  $J'^\mu J'_\mu = J^\mu J_\mu$ . This means that  $\delta(J^\rho J_\rho)/\delta g^{\mu\nu} = 0$ .

In order to find the energy-momentum tensor coming from the kinetic part of the Dirac action

$$S_k = \frac{i}{2} \int d^4x \sqrt{-g} \mathcal{L}_k = \frac{i}{2} \int d^4x \sqrt{-g} \{ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \}, \quad (4.3)$$

we shall consider its variation in  $h_{\mu\nu}$ . By straightforward calculations we obtain

$$\begin{aligned} S'_k = & \frac{i}{2} \int d^4x \sqrt{-g} \left(1 + \frac{h}{2}\right) \left\{ \bar{\psi} \left( \gamma_\nu + \frac{1}{2} h^\rho{}_\nu \gamma_\rho \right) \partial_\mu \psi (g^{\mu\nu} - h^{\mu\nu}) \right. \\ & - \partial_\mu \bar{\psi} \left( \gamma_\nu + \frac{1}{2} h^\rho{}_\nu \gamma_\rho \right) \psi (g^{\mu\nu} - h^{\mu\nu}) + \frac{i}{2} \bar{\psi} \left( \gamma_\nu + \frac{1}{2} h^\rho{}_\nu \gamma_\rho \right) \omega'^{ab}{}_\mu \sigma_{ab} \psi (g^{\mu\nu} - h^{\mu\nu}) \\ & \left. + \frac{i}{2} \omega'^{ab}{}_\mu \bar{\psi} \sigma_{ab} \left( \gamma_\nu + \frac{1}{2} h^\rho{}_\nu \gamma_\rho \right) \psi (g^{\mu\nu} - h^{\mu\nu}) \right\}, \end{aligned} \quad (4.4)$$

where we used the covariant derivative of Dirac spinors in terms of the spin connection,  $\omega^{ab}{}_\mu$ ,

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{2} \omega^{ab}{}_\mu \sigma_{ab} \psi, \quad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \frac{i}{2} \omega^{ab}{}_\mu \bar{\psi} \sigma_{ab}, \quad (4.5)$$

and  $\sigma_{ab} = \frac{i}{2} [\Gamma_a, \Gamma_b]$ . Collecting the terms up to first order in  $h_{\mu\nu}$ , and expressing the spin connection as

$$\omega'^{ab}{}_\mu = \omega^{ab}{}_\mu + \Delta \omega^{ab}{}_\mu,$$

we find

$$S'_k = S_k + \frac{i}{2} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} g_{\mu\nu} \mathcal{L}_k - \frac{1}{2} (\bar{\psi} \gamma_{(\mu} \nabla_{\nu)} \psi - \nabla_{(\nu} \bar{\psi} \gamma_{\mu)} \psi) \right\} h^{\mu\nu} \\ + \frac{i}{2} \int d^4x \sqrt{-g} \{ \bar{\psi} \gamma^\mu \sigma_{ab} \psi + \bar{\psi} \sigma_{ab} \gamma^\mu \psi \} \Delta \omega^{ab}_{\mu}. \quad (4.6)$$

It is easy to verify that  $\Delta \omega^{ab}_{\mu}$  can be written as

$$\Delta \omega^{ab}_{\mu} = \frac{1}{4} \left\{ \nabla_{\beta} \left( e^{a\alpha} e^{b\beta} h_{\alpha\mu} \right) - \nabla_{\beta} \left( e^{b\alpha} e^{a\beta} h_{\alpha\mu} \right) \right\}. \quad (4.7)$$

Replacing this expression into equation (4.6) and performing integration by parts, one can conclude that after some algebra all contributions including  $\Delta \omega^{ab}_{\mu}$  do cancel identically such that the momentum-energy tensor is given in Eq. (2.7).

## Appendix B. Equations of motion

We start from the Dirac equation

$$\frac{i}{a} \Gamma_0 \left( \psi' + \frac{3a'}{2a} \psi \right) = m\psi + 2\theta (\Gamma_5 \Gamma_a \psi) J^a. \quad (4.8)$$

The conjugate can be easily derived, but we do not present it here. Let us remember the definition of the pseudo scalar  $P \equiv i\bar{\psi}\Gamma_5\psi$ , scalar  $S \equiv \bar{\psi}\psi$ , vector current  $V_a = \bar{\psi}\Gamma_a\psi$ , axial current  $J_a = \bar{\psi}\Gamma^5\Gamma_a\psi$ , Lorentz tensor  $L_{ab} = \frac{i}{2}\bar{\psi}[\Gamma_a, \Gamma_b]\psi$ , the factors of  $i$  are chosen such that they are all real.

The first step is to establish the equation of motion for the scalar. In order to do so we multiply (4.8) by  $\bar{\psi}\Gamma_0$  on the left and obtain

$$\frac{i}{a} \left( \bar{\psi}\psi' + \frac{3a'}{2a} \bar{\psi}\psi \right) = m\bar{\psi}\Gamma_0\psi + 2\theta (\bar{\psi}\Gamma_0\Gamma_5\Gamma_a\psi) J^a. \quad (4.9)$$

Since  $\Gamma_5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$  we have that  $\Gamma_0\Gamma_5 = i\Gamma_1\Gamma_2\Gamma_3$  and  $\Gamma_0\Gamma_5\Gamma_a = i\Gamma_1\Gamma_2\Gamma_3\Gamma_a = -\eta_{a0}\Gamma_5 - i/2\epsilon_{aij}\Gamma^i\Gamma^j$ . Thus Eq. (4.9) becomes

$$\frac{i}{a} \left( \bar{\psi}\psi' + \frac{3a'}{2a} \bar{\psi}\psi \right) = mV_0 + 2\theta \left( iJ_0P - \frac{1}{2}\epsilon_{aij}L^{ij}J^a \right), \quad (4.10)$$

while its conjugate gives

$$-\frac{i}{a} \left( \bar{\psi}'\psi + \frac{3a'}{2a} \bar{\psi}\psi \right) = mV_0 + 2\theta \left( -iJ^0P - \frac{1}{2}\epsilon_{aij}L^{ij}J^a \right). \quad (4.11)$$

By taking the imaginary part of (4.10) we get the conservation law

$$\frac{1}{2a^4} (a^3 S)' = 2\theta J_0 P. \quad (4.12)$$

By taking the real part of (4.10) we get

$$\frac{i}{2a} (\bar{\psi}\psi' - \bar{\psi}'\psi) = mV_0 - \theta \epsilon_{aij} L^{ij} J^a. \quad (4.13)$$

Now we look at the pseudo scalar, by contracting (4.8) with  $\bar{\psi}\Gamma_5\Gamma_0$  we obtain:

$$\frac{i}{a} \left( \bar{\psi}\Gamma_5\psi' + \frac{3a'}{2a}\bar{\psi}\Gamma_5\psi \right) = m\bar{\psi}\Gamma_5\Gamma_0\psi + 2\theta (\bar{\psi}\Gamma_5\Gamma_0\Gamma_5\Gamma_a\psi) J^a. \quad (4.14)$$

Using  $\Gamma_5^2 = 1$  and  $\bar{\psi}\Gamma_0\Gamma_a\psi = \bar{\psi}\eta_{0a}\psi - iL_{0a}$ , the RHS reads  $mJ_0 - 2\theta (J_0S - iL_{0a}J^a)$ . Taking the real part we obtain

$$\frac{1}{2a^4} (a^3P)' = mJ_0 - 2\theta J_0S. \quad (4.15)$$

We now consider the components of the axial vector by contracting (4.8) with  $\bar{\psi}\Gamma_5$

$$\frac{i}{a} \left( \bar{\psi}\Gamma_5\Gamma_0\psi' + \frac{3a'}{2a}\bar{\psi}\Gamma_5\Gamma_0\psi \right) = m\bar{\psi}\Gamma_5\psi + 2\theta (\bar{\psi}\Gamma_5\Gamma_5\Gamma_a\psi) J^a. \quad (4.16)$$

The RHS is  $-imP + 2\theta V_aJ^a$ , thus the imaginary part gives

$$\frac{1}{2a^4} (a^3J_0)' = -mP. \quad (4.17)$$

We now look at the spatial components of the axial vector by contracting (4.8) with  $\bar{\psi}\Gamma_5\Gamma_i\Gamma_0$ :

$$\frac{i}{a} \left( \bar{\psi}\Gamma_5\Gamma_i\psi' + \frac{3a'}{2a}\bar{\psi}\Gamma_5\Gamma_i\psi \right) = m\bar{\psi}\Gamma_5\Gamma_i\Gamma_0\psi + 2\theta (\bar{\psi}\Gamma_5\Gamma_i\Gamma_0\Gamma_5\Gamma_a\psi) J^a. \quad (4.18)$$

Since  $\Gamma_5\Gamma_i\Gamma_0 = -i/2\epsilon_{ijk}\Gamma^j\Gamma^k$ , and  $\Gamma_i\Gamma_0\Gamma_a = \eta_{0a}\Gamma_i - \eta_{ia}\Gamma_0 + i\epsilon_{iab}\Gamma_5\Gamma^b$ , the RHS reads  $-1/2m\epsilon_{ijk}L^{jk} + 2\theta (V_iJ_0 - J_iV_0 + i\epsilon_{iab}J^aJ^b)$ . Thus taking the imaginary part we get:

$$\frac{1}{a} (a^3J_i)' = 2\theta\epsilon_{iab}J^aJ^b = 0. \quad (4.19)$$

We now consider the time component vector current by contracting (4.8) with  $\bar{\psi}$ :

$$\frac{i}{a} \left( \bar{\psi}\Gamma_0\psi' + \frac{3a'}{2a}\bar{\psi}\Gamma_0\psi \right) = m\bar{\psi}\psi + 2\theta (\bar{\psi}\Gamma_5\Gamma_a\psi) J^a. \quad (4.20)$$

the RHS reads  $mS + 2\theta J_aJ^a$  which is real, thus we obtain

$$\frac{1}{a^4} (a^3V_0)' = 0, \quad (4.21)$$

while

$$\frac{i}{a} (\bar{\psi}\Gamma_0\psi' - \bar{\psi}'\Gamma_0\psi) = mS + 2\theta J_aJ^a. \quad (4.22)$$

The equation for the space component of vector current by contracting (4.8) with  $\bar{\psi}\Gamma_i\Gamma_0$ :

$$\frac{i}{a} \left( \bar{\psi}\Gamma_i\psi' + \frac{3a'}{2a}\bar{\psi}\Gamma_i\psi \right) = m\bar{\psi}\Gamma_i\Gamma_0\psi + 2\theta (\bar{\psi}\Gamma_i\Gamma_0\Gamma_5\Gamma_a\psi) J^a. \quad (4.23)$$

Now  $\Gamma_i\Gamma_0\Gamma_5\Gamma_a = -\Gamma_i\Gamma_0\Gamma_a\Gamma_5 = -\eta_{0a}\Gamma_i\Gamma_5 + \eta_{ia}\Gamma_0\Gamma_5 - i\epsilon_{aij}\Gamma^j$ , thus the RHS is  $-imL_{i0} + 2\theta i\epsilon_{iaj}J^aV^j$ . Thus we have the evolution equation

$$\frac{1}{2a^4} (a^3V_i)' = mL_{0i} + 2\theta\epsilon_{iaj}J^aV^j \quad (4.24)$$

while

$$\frac{i}{a} (\bar{\psi} \Gamma_i \psi' - \bar{\psi}' \Gamma_i \psi) = 0. \quad (4.25)$$

The equation for the space-time component of Lorentz tensor is obtained by contracting (4.8) with  $\bar{\psi} \Gamma_i$ :

$$\frac{i}{a} \left( \bar{\psi} \Gamma_i \Gamma_0 \psi' + \frac{3a'}{2a} \bar{\psi} \Gamma_i \Gamma_0 \psi \right) = m \bar{\psi} \Gamma_i \psi + 2\theta (\bar{\psi} \Gamma_i \Gamma_5 \Gamma_a \psi) J^a. \quad (4.26)$$

Since  $\Gamma_i \Gamma_5 \Gamma_a = -\eta_{ia} \Gamma_5 - i\epsilon_{iab} \Gamma^0 \Gamma^b + i/2 \eta_{a0} \epsilon_{ijk} \Gamma^j \Gamma^k$  the RHS is  $mV_i + 2\theta(iJ_i P - \epsilon_{iab} L^{0b} J^a + 1/2 \epsilon_{ijk} L^{jk} J_0)$ . Taking the real part of the equation we obtain

$$\frac{1}{2a^4} (a^3 L_{i0})' = mV_i - 2\theta \epsilon_{iab} J^a L^{0b} + \theta J_0 \epsilon_{ijk} L^{jk}. \quad (4.27)$$

The equation for the space components of the Lorentz tensor is obtained by contracting (4.8) with  $\bar{\psi} \Gamma_i \Gamma_j \Gamma_0$ :

$$\frac{i}{a} \left( \bar{\psi} \Gamma_i \Gamma_j \psi' + \frac{3a'}{2a} \bar{\psi} \Gamma_i \Gamma_j \psi \right) = m \bar{\psi} \Gamma_i \Gamma_j \Gamma_0 \psi + 2\theta (\bar{\psi} \Gamma_i \Gamma_j \Gamma_0 \Gamma_5 \Gamma_a \psi) J^a. \quad (4.28)$$

Using that  $\Gamma_i \Gamma_j \Gamma_0 = \eta_{ij} \Gamma_0 - i\epsilon_{ijk} \Gamma_5 \Gamma^k$  and that

$$\Gamma_i \Gamma_j \Gamma_0 \Gamma_5 \Gamma_a = \frac{i}{2} (\eta_{ai} \epsilon_{jbc} - \eta_{aj} \epsilon_{ibc}) \Gamma^b \Gamma^c - i\eta_{a0} \epsilon_{ijk} \Gamma^0 \Gamma^k + i\epsilon_{ija} - \eta_{0a} \eta_{ij} \Gamma_5 - \frac{i}{2} \eta_{ij} \epsilon_{akl} \Gamma^k \Gamma^l,$$

we obtain, antisymmetrizing in  $i$  and  $j$ ,

$$\frac{1}{2a^4} (a^3 L_{ij})' = 2\theta \left( \frac{1}{2} J_i \epsilon_{jab} L^{ab} - \frac{1}{2} J_j \epsilon_{iab} L^{ab} - J_0 \epsilon_{ijk} L^{0k} \right). \quad (4.29)$$

Contracting this with  $\epsilon^{ijk}/2$  gives

$$\frac{1}{2a^4} (a^3 L^k)' = 2\theta \left( \epsilon^{ijk} J_i L_j + J_0 L^{0k} \right). \quad (4.30)$$

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